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Asymptotics of the heat kernel on rank-1 locally symmetric spaces

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Abstract. We consider the heat kernel (and the zeta function) associated with Laplace-type operators acting on a general irreducible rank-1 locally symmetric space X. The set of Minakshisundaram–Pleijel coefficients $\{A_k(X)\}_{k=0}^{\infty}$ in the short-time asymptotic expansion of the kernel is calculated explicitly.

1. Introduction

In the theory of quantum fields on curved background spaces, the short-time expansion of the heat kernel plays an extremely important role. In particular situations, for example, the coefficients in the expansion control the one-loop divergences of the effective action, and related quantities such as the stress energy momentum tensor. Some of these coefficients have been determined and appear in the physics and mathematical literature. Note [1–6] for closed Riemannian manifolds and [7, 8] for manifolds with a smooth boundary. The literature on these matters is vast.

In [1–3], R Miatello studies the case of a closed locally symmetric rank-1 manifold X, using the representation theory of the group of isometries of X. We consider the same case in the present paper, but we use the spectral zeta function of X. By our approach we determine the expansion coefficients immediately and explicitly (essentially in one step), given the results of [9]. Recently the topological Casimir energy [10], the one-loop effective action, and the multiplicative and conformal anomaly [11, 12] associated with Laplace-type operators on X, and their product, have also been analysed by use of the spectral zeta function.

The paper is organized as follows. In section 2 we define the spectral zeta function $\zeta_\Gamma(s;\chi)$ of X corresponding to a finite-dimensional representation χ of the fundamental group Γ of X. The residues of $\zeta_\Gamma(s;\chi)$ and special values of this zeta function, which relate to the expansion coefficients, are provided by theorems 2.1 and 2.2. In section 3 we consider the asymptotic expansion of the heat kernel (as $t \to 0^+$), and compute all the expansion coefficients in closed form in the main theorem, theorem 3.1. Section 5 contains some remarks in summary. We include an appendix with information supplementary to theorems 2.1, 2.2 and 3.1.

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2. The spectral zeta function

We shall be working with an irreducible rank-1 symmetric space $M = G/K$ of non-compact type. Thus G will be a connected non-compact simple split rank-1 Lie group with a finite centre and $K \subset G$ will be a maximal compact subgroup [13]. Let $\Gamma \subset G$ be a discrete, co-compact torsion free subgroup. Then $X = X_\Gamma = \Gamma\backslash M$ is a compact Riemannian manifold with fundamental group Γ; namely X is a compact locally symmetric space. Given a finite-dimensional unitary representation χ of Γ there is the corresponding vector bundle $V_\chi \mapsto X$ over X given by $V_\chi = \Gamma\backslash(M \otimes F_\chi)$, where F_χ (the fibre of V_χ) is the representation space of χ and where Γ acts on $M \otimes F_\chi$ by the rule $\gamma \cdot (m, f) = (\gamma \cdot m, \chi(\gamma) f)$ for $(\gamma, m, f) \in (\Gamma \otimes M \otimes F_\chi)$. Let Δ_Γ be the Laplace–Beltrami operator of X acting on smooth sections of V_χ; we obtain Δ_Γ by projecting the Laplace–Beltrami operator of M (which is G-invariant and thus Γ-invariant) to X. As X is compact we can consider the spectrum $\{\lambda_j = \lambda_j(\chi), n_j = n_j(\chi)\}_{j=0}^{\infty}$ of $-\Delta_\Gamma$, where n_j is the (finite) multiplicity of the eigenvalue λ_j. We use the minus preceding Δ_Γ to have the $\lambda_j \geqslant 0 : 0 = \lambda_0 < \lambda_1 < \lambda_2 \ldots$; $\lim_{j\to\infty} \lambda_j = \infty$.

The spectral zeta function $\zeta_\Gamma(s; \chi)$ of X_Γ of Minakshisundaram–Pleijel type [14], which we shall consider is defined by

$$\zeta_\Gamma(s; \chi) = \sum_{j=1}^{\infty} \frac{n_j(\chi)}{\lambda_j(\chi)^s} \tag{2.1}$$

for $\mathrm{Re}\, s \gg 0$. $\zeta_\Gamma(s; \chi)$ is a holomorphic function on the domain $\mathrm{Re}\, s > d/2$, where d is the dimension of M, and by general principles $\zeta_\Gamma(s; \chi)$ admits a meromorphic continuation to the full complex plane $\mathbb{C}$. However, since the manifold X_Γ is quite special it is desirable to have the meromorphic continuation of $\zeta_\Gamma(s; \chi)$ in an explicit form, for example in terms of the structure of G and Γ. Using the Selberg trace formula and the K-spherical harmonic analysis of G, we have obtained such a form in [9]; also see [10, 15]. In particular we can obtain the residues of $\zeta_\Gamma(s; \chi)$, and compute the special values $\zeta_\Gamma(-n; \chi)$, $n = 0, 1, 2 \ldots$—results which play a decisive role in the present work. To state these results we introduce further notation.

Up to local isomorphism we can represent $M = G/K$ by the following quotients:

$$M = \begin{bmatrix} SO_1(n,1)/SO(n) & (I) \\ SU(n,1)/U(n) & (II) \\ SP(n,1)/(SP(n) \otimes SP(1)) & (III) \\ F_{4(-20)}/\mathrm{Spin}(9) & (IV) \end{bmatrix} \tag{2.2}$$

where $d = n$, $2n$, $4n$, 16, respectively. We shall need the real number ρ_0 which corresponds to $\frac{1}{2}$ the sum of the positive real restricted roots of G with respect to a nilpotent factor in an Iwasawa decomposition of G. ρ_0 is given by $\rho_0 = (n-1)/2, n, 2n+1, 11$ respectively in the cases (I)–(IV). For details on these matters the reader may consult [13], and also the Appendix in [10].

The spherical harmonic analysis on M is controlled by Harish–Chandra's Plancherel density $|C(r)|^{-2}$, a function on the real numbers $\mathbb{R}$, computed by Miatello [1–3], and others, in the rank-1 case we are considering. We choose a normalization of the Haar measure on G, however, which differs from that of [1–3]; see [9]. For a suitable constant C_G depending only on G, and for a suitable even polynomial $P(r)$ of degree $d-2$ for $G \neq SO_1(n, 1)$ with n odd, and of degree $d-1=2m$ for $G=SO_1(2m+1,1)$, $|C(r)|^{-2}$ is given by

$$|C(r)|^{-2}=\left[\begin{array}{lll} C_G\pi r P(r)\tanh(\pi r) & \text{for}\quad G=SO_1(2m,1) & \\ C_G\pi r P(r)\tanh(\pi r/2) & \text{for}\quad G=SU(n,1) & n\ \text{odd} \\ & \text{or}\quad G=SP(n,1) & F_{4(-20)} \\ C_G\pi r P(r)\coth(\pi r/2) & \text{for}\quad G=SU(n,1) & n\ \text{even} \\ C_G\pi P(r) & \text{for}\quad G=SO_1(2m+1,1) & \end{array}\right]. \tag{2.3}$$

The value of C_G and the explicit form of $P(r)$ are given in the appendix. For real hyperbolic space $M=SO_1(2m,1)/SO(2m)$ of even dimension $2m$, for example, $P(r)$ is given by

$$P(r)=\prod_{j=0}^{m-2}\left[r^2+\frac{(2j+1)^2}{4}\right]. \tag{2.4}$$

The coefficients of $P(r)$ will be denoted by a_{2j}:

$$\begin{aligned} P(r)&=\sum_{j=0}^{d/2-1}a_{2j}r^{2j} \qquad \text{for}\quad G\neq SO_1(2m+1,1)\\ &=\sum_{j=0}^{m}a_{2j}r^{2j} \qquad \text{for}\quad G=SO_1(2m+1,1). \end{aligned} \tag{2.5}$$

We denote by $\mathrm{Vol}(\Gamma\backslash G)$ the G-invariant volume of $\Gamma\backslash G$ induced by the Haar measure on G.

As pointed out earlier, the explicit meromorphic structure of the zeta function $\zeta_\Gamma(s;\chi)$ of (2.1) is worked out in [9] in terms of the spherical harmonic analysis of G and Γ-structure; see theorems 4.2, 5.1 there; also compare theorems 5.2, equation (6.1), and theorem 6.9 of [10]. In particular, apart from the case $G=SO_1(n,1)$ with n odd (a case which we treat separately), $\zeta_\Gamma(s;\chi)$ is holomorphic except for possibly simple poles at $s=1,2,\ldots,d/2$. By theorem 5.1 of [9], or by the results stated in [10] we can compute the residues at these points $s=1,2,\ldots,d/2$. The results are the following, where we omit the cotangent case, which will be treated in section 4.

Theorem 2.1. *Apart from the cases $SO_1(\ell,1)$, $SU(q,1)$ with ℓ odd and q even, the residue of $\zeta_\Gamma(s;\chi)$ at $s=m$ (for m an integer, $1\leqslant m\leqslant d/2$) equals*

$$\frac{1}{4}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G\sum_{j=0}^{d/2-m}(-1)^j\binom{m+j-1}{j}\rho_0^{2j}a_{2(m+j-1)} \tag{2.6}$$

given the preceding notation. Also for $n=1,2,\ldots,$

$$\begin{aligned}\zeta_\Gamma(-n;\chi)=\frac{1}{4}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G&\left[\sum_{j=0}^{d/2-1}\frac{(-1)^{j+1}j!\rho_0^{2(j+n+1)}a_{2j}}{(n+1)(n+2)\cdots(n+j+1)}\right.\\ &\left.+2\sum_{j=0}^{d/2-1}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!}\rho_0^{2(n-k)}b_{k+1}(j)a_{2j}\right]\end{aligned} \tag{2.7}$$

where

$$b_p(j)\overset{def}{=}[2^{1-2(p+j)}-1]\left[\frac{\pi}{a(G)}\right]^{2(p+j)}\frac{(-1)^jB_{2(p+j)}}{2(p+j)[(p-1)!]} \tag{2.8}$$

for $p=1,2,\ldots,$ B_r the rth Bernoulli number, and for

$$a(G)\overset{def}{=}\left[\begin{array}{lll}\pi & \text{if}\quad G=SO_1(\ell,1) & \text{with } \ell \text{ even}\\ \frac{\pi}{2} & \text{if}\quad G=SU(q,1) & \text{with } q \text{ odd}\\ & \text{or}\quad G=SP(\ell,1) & \text{any } \ell,\ F_{4(-20)}\end{array}\right]. \tag{2.9}$$

$\zeta_\Gamma(0;\chi)=-n_0(\chi)+$ *(the right-hand side of equation (2.7) evaluated at $n=0$).*

Now we consider the case $G = SO_1(\ell, 1)$ with ℓ odd. By the results of [9], for $G = SO_1(2n+1, 1)$ $\zeta_\Gamma(s;\chi)$ has at most a simple pole at the points $s = d/2 - k$, $k = 0, 1, 2, \ldots$. Moreover, we have the following.

Theorem 2.2. *For $G = SO_1(2n+1, 1)$ the residue of $\zeta_\Gamma(s;\chi)$ at $s = d/2 - k$ (where $d/2 = n + \frac{1}{2}$, $k = 0, 1, 2, \ldots$) equals*

$$\frac{1}{4}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G \sum_{j=0}^{n} \frac{(-1)^{j+n+k}\rho_0^{2(j+k-n)}\Gamma(j+\frac{1}{2})a_{2j}}{(j-n+k)!\Gamma(n+\frac{1}{2}-k)} \tag{2.10}$$

for $k \geqslant n$, and equals

$$\frac{1}{4}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G \sum_{j=0}^{k} \frac{(-1)^{j}\rho_0^{2j}\Gamma(n-k+j+\frac{1}{2})a_{2(n-k+j)}}{j!\Gamma(n+\frac{1}{2}-k)} \tag{2.11}$$

for $0 \leqslant k < n$. Here $\rho_0 = n$. Also $\zeta_\Gamma(0;\chi) = -n_0(\chi)$, whereas $\zeta_\Gamma(-k;\chi) = 0$ for $k = 1, 2, \ldots$.

In theorems 2.1 and 2.2 the constant C_G is given in the appendix.

3. The heat kernel coefficients

The object of interest is the heat kernel $\omega_\Gamma(t;\chi)$ defined for $t > 0$ by

$$\omega_\Gamma(t;\chi) = \sum_{j=0}^{\infty} n_j(\chi)\mathrm{e}^{-\lambda_j(\chi)t}. \tag{3.1}$$

If h_t is the fundamental solution of the heat equation on M, then h_t and $\omega_\Gamma(t;\chi)$ are related by the Selberg trace formula (cf [9])

$$\omega_\Gamma(t;\chi) = \chi(1)\mathrm{Vol}(\Gamma\backslash G)h_t(1) + \theta_\Gamma(t;\chi) \tag{3.2}$$

where the theta function $\theta_\Gamma(t;\chi)$ is given by equation (4.18) of [9] (for $b = 0$ there) and where

$$h_t(1) = \frac{1}{4\pi}\mathrm{e}^{-\rho_0^2 t}\int_{\mathbb{R}} \mathrm{e}^{-r^2 t}|C(r)|^{-2}\,\mathrm{d}r. \tag{3.3}$$

We shall *not* need the result (3.2). Our goal is to compute explicitly all of the coefficients $A_k = A_k(\Gamma, \chi)$ in the asymptotic expansion

$$\omega_\Gamma(t;\chi) \simeq (4\pi t)^{-d/2}\sum_{k=0}^{\infty} A_k t^k \qquad \text{as} \quad t \to 0^+. \tag{3.4}$$

Now $\zeta_\Gamma(s;\chi)$ and $\omega_\Gamma(t;\chi)$ are related by the Mellin transform:

$$\zeta_\Gamma(s;\chi) = \frac{\mathfrak{M}[\omega_\Gamma](s)}{\Gamma(s)} = \frac{1}{\Gamma(s)}\int_0^\infty \omega_\Gamma(t;\chi)t^{s-1}\,\mathrm{d}t \qquad \text{for} \quad \mathrm{Re}\, s > \frac{d}{2}. \tag{3.5}$$

Moreover, one knows by abstract generalities (cf [14, 16] for example) that the coefficients A_k are related to residues and special values of $\zeta_\Gamma(s;\chi)$. Namely for m an integer with $1 \leqslant m \leqslant$ d/2, for d even

$$A_{\frac{d}{2}-m} = (4\pi)^{d/2}\Gamma(m) \times [\text{residue of } \zeta_\Gamma(s;\chi) \text{ at } s = m]. \tag{3.6}$$

Also for a positive integer n

$$A_{\frac{d}{2}+n} = \frac{(-1)^n(4\pi)^{d/2}}{n!}\zeta_\Gamma(-n;\chi) \tag{3.7}$$

whereas

$$A_{\frac{d}{2}} = (4\pi)^{d/2}[n_0(\chi) + \zeta_\Gamma(0;\chi)]. \tag{3.8}$$

For $G = SO_1(2n+1,1)$ (the only case in which d is odd) we have for $k = 0, 1, 2, \ldots$

$$A_k = (4\pi)^{d/2}\Gamma\left(\frac{d}{2} - k\right) \times \left[\text{residue of } \zeta_\Gamma(s;\chi) \text{ at } s = \frac{d}{2} - k\right] \tag{3.9}$$

$d/2 = n + \frac{1}{2}$. Hence by equation (3.6)–(3.9), and theorems 2.1, 2.2 we obtain the following main result.

Theorem 3.1. *The heat kernel $\omega_\Gamma(t;\chi)$ in (3.1) admits an asymptotic expansion (3.4). More precisely, given any non-negative integer N one has*

$$\lim_{t\to 0^+}\left[(4\pi t)^{d/2}\omega_\Gamma(t;\chi) - \sum_{k=0}^{N} A_k(\Gamma,\chi)t^k\right]t^{-N} = 0 \tag{3.10}$$

where, apart from the cotangent case in (2.3) (i.e. the case $G = SU(q,1)$ with q even), the coefficients $A_k(\Gamma,\chi) = A_k(X_\Gamma)$ are given as follows.

For all G except $G = SO_1(\ell,1)$, $SU(q,1)$ with ℓ odd and q even

$$A_k(\Gamma,\chi) = (4\pi)^{\frac{d}{2}-1}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G\pi\sum_{\ell=0}^{k}\frac{(-\rho_0^2)^{k-\ell}}{(k-\ell)!}\left[\frac{d}{2} - (\ell+1)\right]!a_{2[\frac{d}{2}-(\ell+1)]}$$

$$\textit{for}\quad 0 \leqslant k \leqslant \frac{d}{2} - 1 \tag{3.11}$$

$$A_{\frac{d}{2}+n}(\Gamma,\chi) = (-1)^n(4\pi)^{\frac{d}{2}-1}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G\pi$$
$$\times\left[\sum_{j=0}^{\frac{d}{2}-1}(-1)^{j+1}\frac{\rho_0^{2(n+1+j)}j!a_{2j}}{(n+1+j)!} + 2\sum_{j=0}^{\frac{d}{2}-1}\sum_{\ell=0}^{n}(-1)^\ell\frac{\rho_0^{2(n-\ell)}}{(n-\ell)!}b_{\ell+1}(j)a_{2j}\right]$$
$$\textit{for}\quad n = 0, 1, 2, \ldots \tag{3.12}$$

where $b_p(j)$ $(p = 1, 2, \ldots)$ and $a(G)$ are given by (2.8) and (2.9).

For $G = SO_1(2n+1,1)$, $k = 0, 1, 2, \ldots$

$$A_k(\Gamma,\chi) = \pi(4\pi)^{n-\frac{1}{2}}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G\sum_{\ell=0}^{\min(k,n)}\frac{(-n^2)^{k-\ell}\Gamma(n-\ell+\frac{1}{2})a_{2(n-\ell)}}{(k-\ell)!} \tag{3.13}$$

or

$$A_k(\Gamma,\chi) = \pi^{3/2}(4\pi)^{n-\frac{1}{2}}\chi(1)\mathrm{Vol}(\Gamma\backslash G)C_G\sum_{\ell=0}^{\min(k,n)}\frac{(-\rho_0^2)^{k-\ell}[2(n-\ell)]!a_{2(n-\ell)}}{(k-\ell)!(n-l)!2^{2(n-\ell)}} \tag{3.14}$$

using that $\Gamma(m+\frac{1}{2}) = \pi^{1/2}(2m)![2^{2m}m!]^{-1}$.

4. The cotangent case

In theorem 3.1 we computed all the Minakshisundaram–Pleijel coefficients $A_k(X_\Gamma)$ for all compact rank-1 space forms X_Γ (up to local isomorphism) with one exception—namely the case $X_\Gamma = \Gamma\backslash G/K$ with $G = SU(q,1)$ where q is even. Here, as indicated in (2.3), the Plancherel density involves the cotangent function, in contrast to the other cases. Thus we call this case the cotangent case, which we now consider to complete our computation.

We assume $G = SU(q,1)$ where now $q \geqslant 2$ is even. The meromorphic structure of $\zeta_\Gamma(s;\chi)$ in (2.1) in this case differs essentially from the case of odd q in its non-singular terms—not the singular terms of $\zeta_\Gamma(s;\chi)$ where information on poles is determined. One therefore has in fact that for an integer m with $1 \leqslant m \leqslant d/2$ the residue of $\zeta_\Gamma(s;\chi)$ at $s = m$ is also given by (2.6), where now $d/2 = q = \rho_0$. Also formula (2.7) holds provided a different definition of the $b_p(j)$ in (2.8) is employed. Namely the proof of (2.7) for $SU(q,1)$ with q even shows that its validity remains provided we now define $b_p(j)$ by

$$b_p(j) = \frac{(-1)^j 2^{2(p+j)} B_{2(p+j)}}{2(p+j)[(p-1)!]} \tag{4.1}$$

for $p = 1, 2, \ldots,\ j = 0, 1, 2, \ldots$. At this point the earlier discussions apply and we may conclude the following.

Theorem 4.1. *Formulae (3.11) and (3.12) also hold for $G = SU(q,1)$ with $q \geqslant 2$ even, where $d/2 = q = \rho_0$, provided that in formula (3.12) definition (2.8) for $b_p(j)$ is replaced by definition (4.1).*

5. Conclusions

Using results [9] on the meromorphic structure of the zeta function of a rank-1 locally symmetric space X, we have obtained in a quick computation all of the Minakshisundaram–Pleijel coefficients (in closed form) in the short-time asymptotic expansion of the heat kernel on X. Our method differs markedly from that of [1–3]. Besides their mathematical interest these coefficients play an important role in quantum loop effects (such as the conformal anomaly), and in field theory, quantum gravity, and cosmology [17, 18].

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Appendix

The constant C_G in equation (2.3) and the Miatello coefficients a_{2j} of the polynomials $P(r)$ in equation (2.5) appear in the statements of theorems 2.1, 2.2 and 3.1. C_G and $P(r)$ for the various rank-1 simple groups G of this paper are given in table A1.

Table A.1.

G	C_G	$P(r)$
$SO_1(n,1), n \geqslant 2$	$\left[2^{2n-4}\Gamma\left(\frac{n}{2}\right)^2\right]^{-1}$	$\prod_{j=0}^{m-2}\left[r^2 + \frac{(2j+1)^2}{4}\right], n = 2m$ $\prod_{j=0}^{m-1}[r^2 + j^2], n = 2m+1$
$SU(n,1), n \geqslant 2$	$[2^{2n-1}\Gamma(n)^2]^{-1}$	$\prod_{j=1}^{n-1}\left[\frac{r^2}{4} + \frac{(n-2j)^2}{4}\right]$
$SP(n,1), n \geqslant 2$	$\left[2^{4n+1}\Gamma(2n)^2\right]^{-1}$	$\left[\frac{r^2}{4} + \frac{1}{4}\right]\prod_{j=3}^{n+1}\left[\frac{r^2}{4} + \left(n - j + \frac{3}{2}\right)^2\right]\left[\frac{r^2}{4} + \left(n - j + \frac{5}{2}\right)^2\right]$
$F_{4(-20)}$	$[2^{21}\Gamma(8)^2]^{-1}$	$\left[\frac{r^2}{4} + \frac{1}{4}\right]\left[\frac{r^2}{4} + \frac{9}{4}\right]\prod_{j=0}^{4}\left[\frac{r^2}{4} + \left(\frac{2j+1}{2}\right)^2\right]$

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